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The Drach superintegrable systems

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Abstract. Two-dimensional degenerate Hamiltonian systems with cubic invariants are considered using the separation of variables method. For the superintegrable Stäckel systems the cubic invariant is rewritten in the new algebro-geometric form, that is far more elementary than the all the known representations. A complete list of all known systems on the plane that admit a cubic invariant is discussed.

1. Introduction

In 1935 Drach carried out the systematic study of integrable Hamiltonian systems of two degrees of freedom with a cubic second invariant [3]. For the search for an additional polynomial integral of motion he applied the Laplace method [8]. This direct approach leads to a complicated set of nonlinear differential equations, whose nonlinearity has no *a priori* restriction.

In [10] similar results were obtained using the Lagrangian formalism and the generalized Noether theorem. In this method integrals of motion are obtained as solutions of another set of second-order partial differential equations. Recently, two-dimensional Hamiltonian systems with cubic invariants were investigated in the framework of the Maupertuis–Jacobi geometrization procedure [7]. In this case we have to solve an over-determined system of second-order differential equations.

Comparing all these approaches we find different final systems of differential equations. In practice it is difficult to solve these complicated equations in general. Usually, we have to use various simplifying additional assumptions. For instance, the Drach ansatz for the Hamilton function H and for the second cubic invariant K

$$\begin{aligned} H &= p_x p_y + U(x,y) \\ K &= 6w(x,y)\left(\frac{\partial H}{\partial x}p_y - p_x\frac{\partial H}{\partial y}\right) - P(p_x,p_y,x,y) \end{aligned} \tag{1.1}$$

yields ten integrable systems. Starting from the fixed polynomial $P(p_x, p_y, x, y)$ the potential $U(x, y)$ and function $w(x, y)$ are obtained as solutions of some system of differential equations [3]. For the another form of the Hamilton function

$$H = p_x^2 + p_y^2 + V(x,y) \tag{1.2}$$

partial solutions of the corresponding differential equations have been studied in [4, 6, 7, 13]. The most complete classification of known results was later brought together by Hietarinta [5]. This list of the known systems with the cubic invariant was extended in [7].

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Obviously, the Hamilton functions (1.1) and (1.2) are related by canonical transformation. Nevertheless, here we consider both these forms of the Hamilton functions separately for an easier comparison of results with known ones.

The aim of this paper is to study the Drach systems and some other degenerate systems on the plane with the integrals of motion cubic in momentum using as an approach the separation of variables, which belongs to the basic methods of classical mechanics. This method permits us to avoid solving differential equations by construction of integrable systems with the second cubic invariant.

Recall that using the Jacobi and Liouville ideas in 1891 Stäckel began a programme dealing the classification of Hamiltonian systems according to their separability or nonseparability, presenting conditions for separability of the Hamilton–Jacobi equation in orthogonal coordinates [12]. For all the Stäckel systems integrals of motion are quadratic polynomials in momentum. Nevertheless, we prove that eight Drach models belong to the Stäckel family of integrable systems [12] and, moreover, seven of them are degenerate systems [10].

The system is called superintegrable or degenerate if the Hamilton function H is in the involution with two integrals of motion I and K, such that

$$\{H, I\} = \{H, K\} = 0 \qquad \{I, K\} = J(H, I, K). \tag{1.3}$$

Initial integrals and the constant of motion $J(H, I, K)$ are generators of the polynomial associative algebra [1, 9], whose defining relations are polynomials of a certain order in generators.

Below we shall consider two-dimensional integrable systems with two quadratic integrals of motion $I_1 = H, I_2 = I$ and with one cubic invariant K. According to the Bernard–Darboux theorem [21] the system with quadratic integrals $I_1,\ I_2$ belongs to the Stäckel family of integrable systems [12]. We prove that for almost all the known models listed in [5, 7] the corresponding equations of motion are linearized on the Lagrangian submanifold $\mathcal{S}_1 \times \mathcal{S}_2$, which is a product of two spheres $\mathcal{S}_{1,2}$. Moreover, for any dynamic splitting on the several spheres we propose the common form for the additional cubic integral K. This representation of the cubic integral may be applied to construct n-dimensional degenerate Stäckel systems as well.

2. The Stäckel systems

The systems associated with the name of Stäckel [12] are holonomic systems on the phase space $\mathbb{R}^{2n}$ equipped with the canonical variables $\{p_j, q_j\}_{j=1}^n$. The nondegenerate $n \times n$ Stäckel matrix $\boldsymbol{S}$, with entries s_{kj} depending only on q_j

$$\det \boldsymbol{S} \neq 0 \qquad \frac{\partial s_{kj}}{\partial q_m} = 0 \qquad j \neq m$$

defines n functionally independent integrals of motion

$$I_k = \sum_{j=1}^n c_{jk}(p_j^2 + U_j) \qquad c_{jk} = \frac{\mathfrak{s}^{kj}}{\det \boldsymbol{S}} \tag{2.1}$$

which are quadratic in momentum. Here $\boldsymbol{C} = [c_{jk}]$ denotes the inverse matrix to $\boldsymbol{S}$ and $\mathfrak{s}^{kj}$ is the cofactor of the element s_{kj}. The common level surface of these integrals

$$M_\alpha = \{z \in \mathbb{R}^{2n} : I_k(z) = \alpha_k, k = 1, \ldots, n\}$$

is diffeomorphic to the n-dimensional real torus and one immediately obtains

$$p_j^2 = \sum_{i=1}^n \alpha_i s_{ij}(q_j) - U_j(q_j). \tag{2.2}$$

For the rational entries of $\boldsymbol{S}$ and rational potentials $U_j(q_j)$ one obtains

$$p_j^2 = \frac{\prod_{i=1}^k (q_j - e_i)}{\varphi_j^2(q_j)} \tag{2.3}$$

where e_i are constants of motion and functions $\varphi_j(q_j)$ depend on coordinate q_j and numerical constants [15]. The Riemann surfaces (2.3) are isomorphic to the canonical hyperelliptic curves

$$\mathcal{C}_j : \mu_j^2 = \prod_{i=1}^k (\lambda - e_i) \qquad \mu_j = \varphi(q_j) p_j \tag{2.4}$$

where the senior degree k of polynomial fixes the genus $g_j = [(k-1)/2]$ of the algebraic curve $\mathcal{C}_j$. Considered together, these curves determine an n-dimensional Lagrangian submanifold in $\mathbb{R}^{2n}$

$$\mathcal{C}^{(n)} : \mathcal{C}_1(p_1, q_1) \times \mathcal{C}_2(p_2, q_2) \times \cdots \times \mathcal{C}_n(p_n, q_n).$$

The Abel transformation linearizes equations of motion on $\mathcal{C}^{(n)}$ using first-kind Abelian differentials on the corresponding algebraic curves [16]. The basis of first-kind Abelian differentials is uniquely related to the Stäckel matrix $\boldsymbol{S}$ [15, 16].

Now let us turn to the superintegrable or degenerate systems in classical mechanics. One of the main examples of the two-dimensional superintegrable systems is the isotropic harmonic oscillator, which has many common properties with the Drach degenerate systems. Recall that for the oscillator the Hamilton function and the second integral of motion look like

$$H = p_1^2 + p_2^2 + q_1^2 + q_2^2 \qquad I = p_1^2 + q_1^2 - p_2^2 - q_2^2.$$

Obviously, the angular momentum

$$K = q_1 p_2 - p_1 q_2 = \frac{1}{2}\left(p_1 \frac{\mathrm{d}p_2}{\mathrm{d}t} - \frac{\mathrm{d}p_1}{\mathrm{d}t} p_2\right) \tag{2.5}$$

is the third integral of motion. Two pairs of quadratic integrals $I_1 = H, I_2 = I$ and $\tilde{I}_1 = H, \tilde{I}_2 = K^2$ are associated with the following Stäckel matrices:

$$\boldsymbol{S} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{and} \qquad \tilde{\boldsymbol{S}} = \begin{pmatrix} 1 & 0 \\ r^{-2} & 1 \end{pmatrix} \qquad r^2 = x^2 + y^2$$

respectively. So, the corresponding equations of motion may be separated into different curvilinear coordinate systems. The Stäckel systems for which the Hamilton–Jacobi equation separates into more than one coordinate system were studied in [1, 2, 9, 11].

For all the known superintegrable Stäckel systems the number of degrees of freedom $n > g$ is always more than the sum of genuses g_j of the corresponding algebraic curves. In this case the number of independent first-kind Abelian differentials is insufficient for the inversion of the Abel–Jacobi map on $\mathcal{C}^{(n)}$.

To construct inversion of this map for the degenerate systems one has to complete a given basis of the differentials to the set of n differentials. We have some freedom in a choice of complimentary differentials and, therefore, we can associate the different Stäckel matrices with one given Hamilton function [16]. Using first-kind Abelian differentials one obtains a superintegrable Stäckel system with quadratic integrals only. Of course, we can try to add the second- and third-kind Abelian differentials, but we do not know such examples.

Below we prove that for all the known degenerate systems with a cubic integral K the number of degrees of freedom $n = 2$ is more then the sum $g = g_1 + g_2 = 0$ of genuses $g_j = 0$ of the associated Riemann surfaces (2.4). The corresponding dynamic is splitting on two spheres

$$\mathcal{C}_{1,2} : \mu^2 = \alpha_{1,2}\lambda^2 + \beta_{1,2}\lambda + \gamma_{1,2} \qquad g_{1,2} = 0 \tag{2.6}$$

where α_j, β_j and γ_j are constants of motion.

In variables $\mu_{1,2}$ (2.4) the additional cubic integral of motion for all the degenerate Drach systems looks like

$$K = \frac{\det \boldsymbol{S}}{s_{21}s_{22}} \left(\mu_1 \frac{d\mu_2}{dt} - \frac{d\mu_1}{dt}\mu_2 \right). \tag{2.7}$$

This generalized 'angular momentum' gives rise to the first-order integrals (2.5) or the third-order polynomials in momentum depending on the Stäckel matrices $\boldsymbol{S}$ and potentials U_j. In our case it will be a cubic integral, which coincides with the initial Drach integral up to a numerical factor.

To consider the nonlinear algebra of integrals of motion for the Drach systems we shall introduce generators $\{N, a, a^\dagger\}$ instead of the two quadratic integrals $I_1 = H, I_2 = I$, one cubic integrals K and the constant of motion J (1.3). Similar to an oscillator these new generators have the following properties:

$$\begin{aligned} &\{N, a\} = a \qquad \{N, a^\dagger\} = -a^\dagger \\ &\{a, a^\dagger\} = \Phi(I_1, I_2) \qquad aa^\dagger = \Psi(I_1, I_2). \end{aligned} \tag{2.8}$$

Here generator $N(I_1, I_2)$, functions $\Psi(I_1, I_2)$ and $\Phi(I_1, I_2)$ depend on the quadratic Stäckel integrals only. Two other generators a and $a^\dagger$ are functions of all three constants of motion I_1, I_2, K, such that

$$K = \rho(I_1, I_2)(a - a^\dagger) \qquad J(I_1, I_2, K) = \frac{a + a^\dagger}{2}.$$

The relations (2.8) recall the deformed oscillator algebra, which is widely used for the superintegrable systems with quadratic integrals of motion [1, 9]. However, instead of the usual quadratic algebra of integrals we obtain more complicated algebras of integrals. This classical Poisson algebra may be used to construct the corresponding quantum integrals of motion and their eigenvalues and eigenfunctions [2].

3. The Drach systems

Let us reproduce Drach's results in his notations

(a)
$$U = \frac{\alpha}{xy} + \beta x^{r_1} y^{r_2} + \gamma x^{r_2} y^{r_1} \qquad \text{where} \quad r_j^2 + 3r_j + 3 = 0 \tag{3.1}$$
$$P = (xp_x - p_y y)^3 \qquad w = x^2y^2/2$$

(b)
$$U = \frac{\alpha}{\sqrt{xy}} + \frac{\beta}{(y - \mu x)^2} + \frac{\gamma(y + \mu x)}{\sqrt{xy}(y - \mu x)^2} \tag{3.2}$$
$$P = 3(xp_x - p_y y)^2(p_x + \mu p_y) \qquad w = xy(y - \mu x)$$

(c)
$$U = \alpha xy + \frac{\beta}{(y - ax)^2} + \frac{\gamma}{(y + ax)^2} \tag{3.3}$$
$$P = 3(xp_x - p_y y)(p_x^2 - a^2 p_y^2) \qquad w = (y^2 - a^2x^2)/2$$

(d)
$$U = \frac{\alpha}{\sqrt{y(x - a)}} + \frac{\beta}{\sqrt{y(x + a)}} + \frac{\gamma x}{\sqrt{x^2 - a^2}} \tag{3.4}$$
$$P = 3p_y[(xp_x - p_y y)^2 - a^2 p_x^2] \qquad w = -y(x^2 - a^2)$$

(e)
$$U = \frac{\alpha}{\sqrt{xy}} + \frac{\beta}{\sqrt{x}} + \frac{\gamma}{\sqrt{y}} \qquad P = 3p_y p_x(xp_x - p_y y) \qquad w = -2xy \tag{3.5}$$

(f)
$$U = \alpha xy + \beta y \frac{2x^2 + c}{\sqrt{x^2 + c}} + \frac{\gamma x}{\sqrt{x^2 + c}} \qquad (3.6)$$
$$P = 3p_y^2(xp_x - yp_y) \qquad w = \frac{(x^2 + c)}{2}$$

(g)
$$U = \frac{\alpha}{(y + 3mx)^2} + \beta(y - 3mx) + \gamma(y - mx)(y - 9mx) \qquad (3.7)$$
$$P = (p_x + 3mp_y)^2(p_x - 3mp_y) \qquad w = -m(y + 3mx)$$

(h)
$$U = \left(y + \frac{mx}{3}\right)^{-2/3} \left[\alpha + \beta\,(y - mx/3) + \gamma\left(y^2 - \frac{14mxy}{3} + \frac{m^2x^2}{9}\right)\right] \qquad (3.8)$$
$$P = \left(p_x - \frac{mp_y}{3}\right)\left(p_x^2 + \frac{10mp_xp_y}{3} + \frac{m^2p_y^2}{9}\right) \qquad w = -m\left(y + \frac{mx}{3}\right)$$

(k)
$$U = \alpha y^{-1/2} + \beta xy^{-1/2} + \gamma x \qquad P = 3p_x^2p_y \qquad w = -y \qquad (3.9)$$

(l)
$$U = \alpha\left(y - \frac{\rho x}{3}\right) + \beta x^{-1/2} + \gamma x^{-1/2}(y - \rho x) \qquad (3.10)$$
$$P = 3p_xp_y^2 + \rho p_y^3 \qquad w = x.$$

Here $\alpha, \beta, \gamma, \mu, \rho, a, c$ and m are arbitrary parameters. To compare with [3, 10] we corrected function w in the case (g) (3.7) and revised potential U in the case (k) (3.9). Namely, this corrected Hamiltonian is in the involution with the initial Drach cubic integral K (1.1).

With an exception of three cases, (a) (3.1), (h) (3.8) and (l) (3.10), other Drach systems are degenerate or superintegrable Stäckel systems. The separated variables associated with the pair of quadratic integrals $\{I_1 = H, I_2\}$ are the Stäckel variables. Equations of motion may be integrated in quadratures [15], but these quadratures depend on the value of quadratic integral I_2. Thus, instead of the solution of the initial Drach problem related to integrals $\{H, K\}$ we can solve the associated problem with quadratic integrals $\{H, I_2\}$.

In case (h) (3.8) we also have the Stäckel systems [17]. It is only in this case (h) (3.8) that the dynamics is split on two tori and the number of degrees of freedom is equal to the sum of genuses $g = n = 2$. The corresponding system is a non-degenerate integrable system.

In the case (l) (3.10) the Hamilton function coincides with the Hamiltonian of the previous Stäckel system (3.9) at $\rho = 0$. Here we shall not consider this generalized Stäckel system at $\rho \neq 0$.

Below we shall consider the Drach integrals (1.1) up to linear transformations of the coordinates and a rescaling of these integrals. This allows us to remove some parameters in the Hamilton functions without loss of generality. To associate the degenerate Drach Hamiltonians with the Stäckel matrices $\boldsymbol{S}$ we can join these systems into four pairs of systems with a common matrix $\boldsymbol{S}$.

3.1. Case (a)

In our previous paper [18], the first Drach system (3.1) has been related to the three-particle periodic Toda lattice in the centre-of-mass frame. Namely, after canonical change of the time $t = q_{n+1}$ and the Hamiltonian $H = p_{n+1}$ at the extended phase space

$$\mathrm{d}\tilde{t} = (xy)^{-1}\,\mathrm{d}t \qquad \tilde{H} = xy(H + \delta)$$

and after further canonical transformation of other variables

$$x = \mathrm{e}^{\frac{q_1 + \mathrm{i}q_2}{2}} \qquad p_x = (p_1 - \mathrm{i}p_2)\mathrm{e}^{-\frac{q_1 + \mathrm{i}q_2}{2}} \qquad y = \mathrm{e}^{\frac{q_1 - \mathrm{i}q_2}{2}} \qquad p_y = (p_1 + \mathrm{i}p_2)\mathrm{e}^{-\frac{q_1 - \mathrm{i}q_2}{2}}$$

the Hamilton function (3.1) becomes

$$\tilde{H} = p_1^2 + p_2^2 + \beta e^{-\frac{1}{2}q_1 - \frac{\sqrt{3}}{2}q_2} + \gamma e^{-\frac{1}{2}q_1 + \frac{\sqrt{3}}{2}q_2} + \delta e^{q_1} + \alpha.$$

This is the Hamiltonian of the tree-particle periodic Toda lattice in the centre-of-mass frame. The separated variables survive at the change of the time. Thus, for the first Drach system we can separate variables and integrate equations of motions in quadratures repeating the calculations for the Toda chain [19].

Later in [13] Thompson considered this system too. In fact, after point transformation

$$x = re^{i\phi} \qquad p_x = \frac{e^{-i\phi}}{2}(p_r - ip_\phi r^{-1}) \qquad y = re^{-i\phi} \qquad p_y = \frac{e^{i\phi}}{2}(p_r + ip_\phi r^{-1})$$

the Drach Hamiltonian H (1.1) looks like

$$H = p_r^2 + \frac{p_\phi^2}{r^2} + U(r, \phi)$$

up to a numerical factor. This particular Hamilton function was studied in [7] and [13]. The special substitution of the potential $U(r, \phi)$ into the Drach equations leads to the following equation:

$$U(r, \phi) = \frac{f(\phi) + f''(\phi)}{r^3} \Rightarrow f'''f'' - 2f''f' - 3f'f = 0$$

introduced in [7, 13]. Of course, the same equation follows from the functional equation on the Toda potential [5].

3.2. Cases (b) and (e)

Put $\mu = 1$ in (3.2). Let us introduce the Stäckel matrix

$$\boldsymbol{S}_{\mathrm{be}} = \begin{pmatrix} q_1^2 & q_2^2 \\ 1 & 1 \end{pmatrix} \tag{3.11}$$

and take the following potentials:

(b) $\qquad U_1 = 2\alpha - \dfrac{\beta - 2\gamma}{q_1^2} \qquad U_2 = -2\alpha - \dfrac{\beta + 2\gamma}{q_2^2}$

(e) $\qquad U_1 = 2\alpha + 2(\beta + \gamma)q_1 \qquad U_2 = -2\alpha - 2(\beta - \gamma)q_2.$

The corresponding Hamilton functions I_1 (2.1) coincide with the Hamilton functions H for the Drach systems (3.2) and (3.5), after the following canonical point transformation:

$$x = \frac{(q_1 - q_2)^2}{4} \qquad p_x = \frac{p_1 - p_2}{q_1 - q_2} \qquad y = \frac{(q_1 + q_2)^2}{4} \qquad p_y = \frac{p_1 + p_2}{q_1 + q_2}.$$

The second integrals of motion I_2 (2.1) are second-order polynomials in momentum. The third independent integrals of motion K are defined by (2.7), where

(b) $\qquad \mu_1 = q_1 p_1 \qquad \mu_2 = q_2 p_2$

(e) $\qquad \mu_1 = p_1 \qquad \mu_2 = p_2.$

From the above definitions we can introduce generators of the nonlinear algebra of integrals (2.8) and verify properties of this algebra

(b) $\qquad N = \dfrac{I_2}{4\sqrt{H}} \qquad a = J + 4\sqrt{H}K \qquad a^\dagger = J - 4\sqrt{H}K$

$$aa^\dagger = 16(4H(\beta + 2\gamma) - (2\alpha + I_2)^2)(4H(\beta - 2\gamma) - (2\alpha - I_2)^2)$$

$$\{a, a^\dagger\} = -256\sqrt{H}(I_2(I_2 - 2\alpha)(I_2 + 2\alpha) - 4H(\beta I_2 - 4\alpha\gamma))$$

and

(e) $$N = \frac{I_2}{2\sqrt{H}} \qquad a = J + 2\sqrt{H}K \qquad a^\dagger = J - 2\sqrt{H}K$$
$$aa^\dagger = -16(H\,(2\alpha + I_2) - (\beta - \gamma)^2)(H\,(2\alpha - I_2) + (\beta + \gamma)^2)$$
$$\{a, a^\dagger\} = -64H^{3/2}(I_2 H - \beta^2 - \gamma^2).$$

3.3. Cases (c) and (g)

Put $a = 1$ in (3.3) and $m = 1/3$ in (3.7). Let us introduce the Stäckel matrix

$$\boldsymbol{S}_{\text{cg}} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{pmatrix} \tag{3.12}$$

and take the following potentials:

$$\begin{aligned} &\text{(c)} \qquad U_1 = \frac{\alpha q_1^2}{4} + \frac{\gamma}{q_1^2} \qquad U_2 = \frac{\alpha q_2}{4} - \frac{\beta}{q_2^2} \\ &\text{(g)} \qquad U_1 = -\frac{\gamma q_1^2}{3} + \frac{\alpha}{q_1^2} \qquad U_2 = -\frac{4\gamma q_2^2}{3} - \beta q_2. \end{aligned} \tag{3.13}$$

The corresponding Hamilton functions I_1 (2.1) coincide with the Hamilton functions H (3.3) and (3.7), after the following canonical point transformation:

$$x = \frac{q_1 - q_2}{2} \qquad p_x = p_1 - p_2 \qquad y = \frac{q_1 + q_2}{2} \qquad p_y = p_1 + p_2.$$

The second integrals of motion I_2 (2.1) are the second-order polynomials in momentum. The third independent integrals K are defined by (2.7), where

(c) $\mu_1 = q_1 p_1 \qquad \mu_2 = q_2 p_2$

(g) $\mu_1 = q_1 p_1 \qquad \mu_2 = p_2.$

Generators and defining relations of the nonlinear algebra of integrals (2.8) look like

(c) $$N = \frac{I_2}{2\sqrt{-\alpha}} \qquad a = J + 2\sqrt{-\alpha}K \qquad a^\dagger = J - 2\sqrt{-\alpha}K$$
$$aa^\dagger = (H^2 + 4H\,I_2 + 4I_2^2 - 4\alpha\gamma)(H^2 - 4H\,I_2 + 4I_2^2 + 4\alpha\beta)$$
$$\{a, a^\dagger\} = -32\sqrt{a}(I_2(H - 2I_2)(H + 2I_2) - \alpha\beta(H + 2I_2) - \alpha\gamma(H - 2I_2))$$

and

(g) $$N = \frac{I_2}{4}\sqrt{\frac{3}{\gamma}} \qquad a = J + 4\sqrt{\frac{\gamma}{3}}K \qquad a^\dagger = J - 4\sqrt{\frac{\gamma}{3}}K$$
$$aa^\dagger = \tfrac{1}{9}(8\gamma H - 16\gamma I_2 + 3\beta^2)(3H^2 + 12H\,I_2 + 12I_2^2 + 16\alpha\gamma)$$
$$\{aa^\dagger\} = 64\left(\frac{\gamma}{3}\right)^{3/2}\left(\frac{(2I_2 + H)(4\gamma H - 24\gamma I_2 + 3b^2)}{4\gamma} - \frac{16\alpha\gamma}{3}\right).$$

3.4. Cases (d) and (f)

Put $a = 1$ in (3.4) and $c = 1$ in (3.6). Let us introduce two Stäckel matrices

$$\boldsymbol{S}_d = \begin{pmatrix} 1 & 1 \\ \frac{1}{q_1^2} & \frac{1}{q_2^2} \end{pmatrix} \qquad \boldsymbol{S}_f = \begin{pmatrix} \frac{1}{q_1} & \frac{1}{q_2} \\ \frac{1}{q_1^2} & \frac{1}{q_2^2} \end{pmatrix} \tag{3.14}$$

and take the following potentials

$$\text{(d)}\qquad U_1 = 2\gamma + \frac{2\sqrt{2}(\alpha+\beta)}{q_1} \qquad U_2 = -2\gamma - \frac{2\sqrt{2}(\alpha-\beta)}{q_2}$$

$$\text{(f)}\qquad U_1 = \frac{\gamma}{2q_1} + \frac{(\alpha+2\beta)}{4} \qquad U_2 = \frac{\gamma}{2q_2} + \frac{(\alpha-2\beta)}{4}.$$

The corresponding Hamilton functions I_1 (2.1) coincide with the Hamilton functions H (3.4) and (3.6) up to a numerical factor, after the following explicit canonical transformations:

$$\text{(d)}\qquad x = \frac{q_1^2+q_2^2}{2q_1q_2} \qquad p_x = \frac{(p_1q_1 - p_2q_2)q_1q_2}{q_1^2-q_2^2}$$

$$y = q_1q_2 \qquad p_y = \frac{p_1q_1+p_2q_2}{2q_1q_2}$$

$$\text{(f)}\qquad x = \frac{q_1-q_2}{2\sqrt{q_1q_2}} \qquad p_x = \frac{2(p_1q_1-p_2q_2)\sqrt{q_1q_2}}{q_1+q_2}$$

$$y = \sqrt{q_1q_2} \qquad p_y = \frac{p_1q_1+p_2q_2}{\sqrt{q_1q_2}}.$$

The second integrals of motion I_2 (2.1) are the quadratic polynomials in momentum. The third independent integrals K are defined by (2.7), where for both systems one obtains

$$\mu_1 = q_1p_1 \qquad \mu_2 = q_2p_2.$$

Generators of the nonlinear algebra of integrals (2.8) are given by

$$N = \sqrt{I_2} \qquad a = J + 2\sqrt{I_2}K \qquad a^\dagger = J - 2\sqrt{I_2}K$$

which have the following properties:

$$\text{(d)}\qquad aa^\dagger = 16(I_2(2\gamma - H) + 2(\alpha+\beta)^2)(I_2(2\gamma - H) - 2(\alpha-\beta)^2)$$

$$\{a, a^\dagger\} = 64\sqrt{I_2}(I_2(2\gamma - H)(2\gamma + H) + 2(\alpha^2+\beta^2)H + 8\alpha\beta\gamma)$$

$$\text{(f)}\qquad aa^\dagger = \tfrac{1}{16}(4I_2(\alpha+2\beta) + (\gamma - 2H)^2)(4I_2(\alpha - 2\beta) + (\gamma+2H)^2)$$

$$\{a, a^\dagger\} = -4\sqrt{I_2}\left((2\beta-\alpha)(2\beta+\alpha)I_2 + \alpha H^2 + 2\beta\gamma H + \frac{\alpha\gamma^2}{4}\right).$$

3.5. *Cases (h) and (k)*

Put $m = 3$ in (3.8). Let us introduce two the Stäckel matrices

$$\boldsymbol{S}_h = \begin{pmatrix} q_1 & q_2 \\ -1 & 1 \end{pmatrix} \qquad \boldsymbol{S}_k = \begin{pmatrix} q_1 & -q_2 \\ 1 & 1 \end{pmatrix} \tag{3.15}$$

and take the following potentials:

$$\text{(h)}\qquad U_1 = \frac{\alpha\gamma - \frac{\beta^2}{16}}{4\gamma} - \frac{2\gamma q_1^3}{9} \qquad U_2 = \frac{\alpha\gamma - \frac{\beta^2}{16}}{4\gamma} - \frac{2\gamma q_2^3}{9}$$

$$\text{(k)}\qquad U_1 = \alpha + \beta q_1 + \frac{\gamma q_1^2}{2} \qquad U_2 = -\alpha + \beta q_2 + \frac{\gamma q_2^2}{2}.$$

The corresponding Hamilton functions I_1 (2.1) coincide with the Hamilton functions H (3.8) and (3.9) up to a numerical factor, after the following explicit canonical transformations:

$$\text{(h)}\qquad x = \frac{p_2 - p_1}{4\sqrt{\gamma}} + \frac{(3q_1+3q_2)^{3/2}}{54} + \frac{\beta}{16\gamma} \qquad p_x = 3\frac{p_1+p_2}{\sqrt{3q_1+3q_2}} + \sqrt{\gamma}(q_1 - q_2)$$

$$y = -\frac{p_2 - p_1}{4\sqrt{\gamma}} + \frac{(3q_1 + 3q_2)^{3/2}}{54} - \frac{\beta}{16\gamma}$$

$$p_y = 3\frac{p_1 + p_2}{\sqrt{3q_1 + 3q_2}} - \sqrt{\gamma}(q_1 - q_2)$$

and

(k) $$x = \frac{q_1 - q_2}{2} \qquad p_x = p_1 - p_2 \qquad y = \frac{(q_1 + q_2)^2}{4} \qquad p_y = \frac{p_1 + p_2}{q_1 + q_2}.$$

Note that in the case (h) (3.8) we used non-point canonical transformation in contrast with other Drach systems.

In the last case (k), (3.9) the integral of motion I_2 (2.1) is a second-order polynomial in momentum. The third independent integral K is defined by (2.7), where

$$\mu_1 = p_1 \qquad \text{and} \qquad \mu_2 = p_2.$$

Generators and defining relations of the nonlinear algebra of integrals (2.8) look like

(k) $$N = \frac{I_2}{\sqrt{-2\gamma}} \qquad a = J + \sqrt{-2\gamma}K \qquad a^\dagger = J - \sqrt{-2\gamma}K$$

$$aa^\dagger = (2\gamma(I_2 + \alpha) + (H + \beta)^2)(2\gamma(I_2 - \alpha) + (H - \beta)^2)$$

$$\{a, a^\dagger\} = -4\gamma\sqrt{-2c}(H^2 + 2\gamma I_2 + \beta^2).$$

In the case (h) (3.8) the second integral of motion I_2 (2.1) is the second-order polynomial in momentum $\{p_1, p_2\}$. However, after the non-point transformation of variables this integral I_2 becomes the $\{p_x, p_y\}$ Drach integral K (3.8) cubic in momentum. The corresponding dynamics is split on two tori and the third-order polynomial (2.7) does not commute with the Hamilton function. Later this system was rediscovered by Holt [6].

4. Other degenerate systems on the plane with a cubic integral of motion

In this section we consider the Stäckel systems on the plane with a cubic integral of motion defined by the following Hamilton function:

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + V(x, y).$$

As above, the corresponding cubic integral will be written in the Drach form (1.1).

On the plane we know four orthogonal systems of coordinates: elliptic, parabolic, polar and Cartesian. Thus, we reproduce all the known results [5,7] in correspondence with the type of the associated Stäckel matrix [15,16].

The systems whose Hamilton functions are separable in Cartesian coordinates:

(A) $$V = \alpha(4x^2 + y^2) + \beta x + \frac{\gamma}{y^2} \tag{4.1}$$

$$P = p_x p_y^2 \qquad -w = \frac{y}{6}$$

(B) $$V = \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2} \tag{4.2}$$

$$P = (xp_y - yp_x)p_x p_y \qquad -w = \frac{xy}{6}$$

(C) $$V = \alpha(x^2 + y^2) + \beta\frac{xy}{(x^2 - y^2)^2} \tag{4.3}$$

$$P = (x\,p_y - yp_x)(p_x^2 - p_y^2) \qquad -w = \frac{x^2 - y^2}{6}$$

(D) $$V = \alpha(9x^2 + y^2) \tag{4.4}$$

$$P = (xp_y - p_x y)p_y^2 \qquad -w = -\frac{y^2}{18}.$$

The systems whose Hamilton functions are separable in parabolic coordinates:

(F) $$V = \left(\alpha + \frac{\beta}{r+x} + \frac{\gamma}{r-x}\right) r^{-1} \qquad r = \sqrt{x^2+y^2} \tag{4.5}$$

$$P = (xp_y - p_x y)^2 p_x \qquad -w = \frac{yr^2}{12}$$

(G) $$V = \left(\alpha + \frac{\beta x}{y^2}\right) r^{-1} \tag{4.6}$$

$$P = (xp_y - p_x y)^2 p_x \qquad -w = \frac{yr^2}{12}$$

(H) $$V = \left(\alpha + \beta\sqrt{r+x} + \gamma\sqrt{r-x}\right) r^{-1} \tag{4.7}$$

$$P = (xp_y - p_x y)\left(2p_x^2 + 2p_y^2 - \frac{\beta}{\sqrt{r+x}} - \frac{\gamma}{\sqrt{r-x}}\right) \qquad -w = -\frac{r^2}{6}.$$

One system with the Hamilton function separable in polar coordinates:

(I) $$V = \alpha + \frac{\beta}{\sqrt{x^2+y^2}} + \frac{\rho}{(\delta x + \gamma y)^2} + \frac{\gamma x - \delta y}{\sqrt{x^2+y^2}(\delta x + \gamma y)^2} \tag{4.8}$$

$$P = (p_x y - p_y x)(\gamma p_x - \delta p_y) \qquad w = \frac{(x^2+y^2)(\delta x + \gamma y)}{12}.$$

This is a new superintegrable system with a cubic integral of motion, which was not obtained in the framework of the direct approach [4–6] or the Maupertuis–Jacobi procedure [7].

Three exceptional systems whose cubic integral of motion K we cannot rewrite in the 'generalized angular momentum' form (2.7):

(K) $$V = \alpha\sqrt{x} \pm \beta\sqrt{y} \qquad P = \frac{\beta}{\alpha} p_x^3 \mp \frac{\alpha}{\beta} p_y^3 \qquad w = \sqrt{xy} \tag{4.9}$$

(L) $$V = \alpha(\sqrt{x} + \beta y) \qquad P = p_x^3 \qquad w = -\frac{\sqrt{x}}{2\beta} \tag{4.10}$$

(M) $$V = f'(\phi) r^{-2} \tag{4.11}$$

$$K = p_\phi^2(\cos\phi p_r - \sin\phi r^{-1} p_\phi) + (2f'(\phi)\cos\phi - f(\phi)\sin(\phi))p_r + (3f'(\phi)\sin\phi + f(\phi)\cos\phi) r^{-1} p_\phi.$$

For the case (M) (4.11) we used the standard polar coordinates $\{r, p_r, \phi, p_\phi\}$ and the function $f(\phi)$ has to satisfy the following equation:

$$f''(3f'\sin\phi + f\cos\phi) + 2f'(2f'\cos\phi - f\sin\phi) = 0.$$

For these exceptional cases the Hamilton functions (4.9)–(4.11) are separable in the Cartesian and polar coordinates, respectively. Note that the corresponding Stäckel potentials $U_{1,2}$ are algebraic or trigonometric functions in separated variables.

4.1. Cartesian coordinates, cases (A)–(D)

Let us introduce the Stäckel matrix

$$\boldsymbol{S}_{(\mathrm{A})-(\mathrm{D})} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix} \tag{4.12}$$

and take the following potentials:

$$
\begin{array}{llll}
\text{(A)} & U_1 = 8\alpha q_1^2 + 2\beta q_1 & U_2 = 2\alpha q_2^2 + \dfrac{2\gamma}{q_2^2} & \\
\text{(B)} & U_1 = 2\alpha q_1^2 + \dfrac{2\beta}{q_1^2} & U_2 = 2\alpha q_2^2 + \dfrac{2\gamma}{q_2^2} & \\
\text{(C)} & U_1 = \dfrac{\alpha q_1^2}{2} - \dfrac{\beta}{4q_1^2} & U_2 = \dfrac{\alpha q_2^2}{2} + \dfrac{\beta}{4q_2^2} & \\
\text{(D)} & U_1 = 18\alpha\, q_1^2 & U_2 = 2\alpha q_2^2. &
\end{array}
\qquad (4.13)
$$

The corresponding Hamilton functions I_1 (2.1) coincide with the Hamilton functions H (4.1), (4.2)–(4.4) if

(A)–(B), (D) $\quad x = q_1 \qquad y = q_2$

or after the following canonical transformation:

$$\text{(C)} \qquad x = \frac{q_1 - q_2}{2} \qquad p_x = p_1 - p_2 \qquad y = \frac{q_1 + q_2}{2} \qquad p_y = p_1 + p_2.$$

The second integrals of motion I_2 (2.1) are second-order polynomials in momentum. The third independent integrals K are calculated by (2.7), where variables

(A) $\quad \mu_1 = p_1 \qquad \mu_2 = q_2 p_2 \qquad$ (B)–(C) $\quad \mu_1 = q_1 p_1 \qquad \mu_2 = q_2 p_2$

determine the left-hand side of the canonical algebraic curves (2.4). In case (D) the variables

$$\text{(D)} \qquad \mu_1 = p_2 q_1 - \frac{p_1 q_2}{3} \qquad \mu_2 = p_2 q_2$$

donot have such a natural algebro-geometric meaning.

Generators and defining relations of the nonlinear algebra of integrals (2.8) look like

$$\text{(A)–(B)} \qquad N = \frac{I_2}{4\sqrt{-2\alpha}} \qquad a = J + 4\sqrt{-2\alpha}K \qquad a^\dagger = J - 4\sqrt{-2\alpha}K$$

such that

$$
\begin{aligned}
\text{(A)} \qquad & aa^\dagger = 4(4\alpha(2I_2 + H) + \beta^2)((2I_2 - H)^2 - 64\alpha\gamma) \\
& \{a, a^\dagger\} = -128\alpha\sqrt{-2\alpha}\left((2I_2 - H)\left(6I_2 + H + \frac{\beta^2}{2\alpha}\right) - 64\alpha\gamma\right) \\
\text{(B)} \qquad & aa^\dagger = ((2I_2 + H)^2 - 64\alpha\beta)((2I_2 - H)^2 - 64\alpha\gamma) \\
& \{a, a^\dagger\} = -16\sqrt{-2\alpha}(((2I_2 - H)^2 - 64\alpha\gamma)(2I_2 + H) \\
& \qquad + ((2I_2 + H)^2 - 64\alpha\beta)(2I_2 - H)).
\end{aligned}
$$

For the two last cases we have

$$
\begin{aligned}
\text{(C)} \qquad & N = \frac{I_2}{2\sqrt{-2\alpha}} \qquad a = J + 2\sqrt{-2\alpha}K \qquad a^\dagger = J - 2\sqrt{-2\alpha}K \\
& aa^\dagger = ((2I_2 - H)^2 - 2\alpha\beta)((2I_2 + H)^2 + 2\alpha\beta) \\
& \{a, a^\dagger\} = -8\sqrt{-2\alpha}(((2I_2 - H)^2 - 2\alpha\beta)(2I_2 + H) \\
& \qquad + ((2I_2 + H)^2 + 2\alpha\beta)(2I_2 - H))
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(D)} \qquad & N = \frac{I_2}{6\sqrt{-2\alpha}} \qquad a = J + 6\sqrt{-2\alpha}K \qquad a^\dagger = J - 6\sqrt{-2\alpha}K \\
& aa^\dagger = -4(2I_2 - H)^3(2I_2 + H) \\
& \{a, a^\dagger\} = -96\sqrt{-2\alpha}(2I_2 - H)^2(4I_2 + H).
\end{aligned}
$$

For case (D) (4.4) the quantum counterpart of this cubic deformed oscillator algebra has been used to study the corresponding quantum superintegrable system [1].

4.2. Parabolic coordinates, cases (F)–(H)

Let us introduce two Stäckel matrices

$$\boldsymbol{S}_{(\mathrm{F}),(\mathrm{G})} = \begin{pmatrix} 1 & 1 \\ q_1^{-1} & q_2^{-1} \end{pmatrix} \qquad \boldsymbol{S}_{(\mathrm{H})} = \begin{pmatrix} q_1^2 & q_2^2 \\ 1 & 1 \end{pmatrix} \tag{4.14}$$

and take the following potentials:

$$\begin{aligned} &(\mathrm{F}) \qquad U_1 = -\frac{\alpha}{2q_1} - \frac{\beta}{2q_1^2} \qquad U_2 = \frac{\alpha}{2q_2} - \frac{\gamma}{2q_2^2} \\ &(\mathrm{G}) \qquad U_1 = \frac{\alpha}{2q_1} - \frac{\beta}{4q_1^2} \qquad U_2 = -\frac{\alpha}{2q_2} + \frac{\beta}{4q_2^2} \\ &(\mathrm{H}) \qquad U_1 = -4\alpha - 8\sqrt{2}\beta q_1 \qquad U_2 = 4\alpha + 8\sqrt{-2}\gamma q_2. \end{aligned} \tag{4.15}$$

The corresponding Hamilton functions I_1 (2.1) coincide with the Hamilton functions H (4.1), (4.2) and (4.3) after the following canonical point transformations:

$$(\mathrm{F}), (\mathrm{G}) \qquad x = q_1 + q_2 \qquad p_x = \frac{p_1 q_2 - p_2 q_2}{q_1 - q_2}$$

$$y = 2\sqrt{-q_1 q_2} \qquad p_y = \frac{(p_1 - p_2)\sqrt{-q_1 q_2}}{q_1 - q_2}$$

and

$$(\mathrm{H}) \qquad x = q_1^2 + q_2^2 \qquad p_x = \frac{1}{2}\frac{p_1 q_1 - p_2 q_2}{q_1^2 - q_2^2}$$

$$y = 2\mathrm{i} q_1 q_2 \qquad p_y = \frac{\mathrm{i}}{2}\frac{p_1 q_2 - p_2 q_1}{q_1^2 - q_2^2}.$$

The second integrals of motion I_2 (2.1) are second-order polynomials in momentum. The third independent integrals K are calculated by (2.7), where variables

$$(\mathrm{F}), (\mathrm{G}) \qquad \mu_1 = q_1 p_1 \qquad \mu_2 = q_2 p_2 \qquad (\mathrm{H}) \quad \mu_1 = p_1 \qquad \mu_2 = p_2$$

define the left-hand side of the canonical algebraic curves (2.4).

For all these cases (F), (G) and (H) the generators and defining relations of the nonlinear algebra of integrals (2.8) look like

$$(\mathrm{F})\text{–}(\mathrm{H}) \qquad N = \frac{I_2}{2\sqrt{H}} \qquad a = J + 2\sqrt{H}K \qquad a^\dagger = J - 2\sqrt{H}K$$

such that

$$\begin{aligned} (\mathrm{F}) \qquad & aa^\dagger = \tfrac{1}{16}(8\beta H - (2I_2 + \alpha)^2)(8\gamma H - (2I_2 - \alpha)^2) \\ & \{a, a^\dagger\} = -2\sqrt{H}(I_2(2I_2 + \alpha)(2I_2 - \alpha) - 2H(\beta(2I_2 - \alpha) + \gamma(2I_2 + \alpha))) \\ (\mathrm{G}) \qquad & aa^\dagger = -\tfrac{1}{16}(4\beta H + (2I_2 + \alpha)^2)(4\beta H - (2I_2 - \alpha)^2) \\ & \{a, a^\dagger\} = 2\sqrt{H}(I_2(2I_2 + \alpha)(2I_2 - \alpha) - 2\alpha\beta H) \\ (\mathrm{H}) \qquad & aa^\dagger = 16(H(I_2 - 4\alpha) + 32\gamma^2)(H(I_2 + 4\alpha) - 32\beta^2) \\ & \{a, a^\dagger\} = -64H^{3/2}(HI_2 - 16\beta^2 + 16\gamma^2). \end{aligned}$$

4.3. Polar coordinates, case (I)

Let us introduce the Stäckel matrix

$$\boldsymbol{S}_{(\mathrm{I})} = \begin{pmatrix} 1 & 0 \\ q_1^{-2} & 1 \end{pmatrix} \tag{4.16}$$

and take the following potentials:

$$\text{(I)} \qquad U_1 = \alpha + \frac{\beta}{q_1} \qquad U_2 = \frac{\gamma\cos(q_2) - \delta\sin(q_2) + \rho}{(\delta\cos(q_2) + \gamma\sin(q_2))^2}. \tag{4.17}$$

The corresponding Hamilton function I_1 (2.1) coincides with the Hamilton function H (4.8) if $q_1 = r$ and $q_2 = \phi$ are the standard polar coordinates on the plane. The second integrals of motion I_2 (2.1) are the second-order polynomials in momentum. The third independent integrals K are calculated by (2.7), where variables

$$\text{(I)} \qquad \mu_1 = p_1 \qquad \mu_2 = p_2(\delta\cos(q_2) + \gamma\sin(q_2))$$

define the canonical algebraic curves (2.4).

For $\delta = 1$ and $\gamma = 0$ the generators and defining relations of the nonlinear algebra of integrals (2.8) look like

$$\begin{aligned} \text{(H)} \qquad & N = -\sqrt{-I_2} \qquad a = J + 2\sqrt{-I_2}K \qquad a^\dagger = J - 2\sqrt{-I_2}K \\ & aa^\dagger = (4HI_2 - 4\alpha I_2 + \beta^2)(4I_2^2 - 4\rho I_2 + 1) \\ & \{a, a^\dagger\} = -8\sqrt{-I_2}((2I_2 - \rho)\beta^2 + (12I_2^2 - 8\rho I_2 + 1)(H - \alpha)). \end{aligned} \tag{4.18}$$

This system is not contained in the list of known integrable systems [5,7]. For the cases (I) (4.8) and (M) (4.11) we have a common leading part P of the cubic integrals K. However, for the case (M) we cannot rewrite the cubic integral in the 'generalized angular momentum' form.

5. The Lax representation

In [15, 16] we proposed some constructions of the 2×2 Lax matrices for the Stäckel systems with homogeneous Stäckel matrices [16] and with uniform potentials $U_j = U$. The Drach–Stäckel systems are not contained in this subset of the Stäckel systems. Nevertheless, we could construct the 2×2 Lax matrices for these systems using various coverings [15] of the initial spheres $\mathcal{C}_{1,2}$ (2.6).

Here we consider the 4×4 Lax matrices for some Drach systems by using canonical transformations of the extended phase space, which induce transformations of the Lax matrices [16, 18]. Recall that if the Stäckel matrices $\boldsymbol{S}$ and $\tilde{\boldsymbol{S}}$ are distinguished in the first row only, the corresponding Stäckel systems are related by a canonical change of the time $q_{n+1} = t$ and conjugated momentum $p_{n+1} = -H$ [16]

$$t \mapsto \tilde{t} \qquad \mathrm{d}\tilde{t} = \frac{\det \tilde{\boldsymbol{S}}}{\det \boldsymbol{S}}\,\mathrm{d}t \qquad H \mapsto \tilde{H} = \frac{\det \boldsymbol{S}}{\det \tilde{\boldsymbol{S}}} H. \tag{5.1}$$

Thus, starting with the Stäckel systems related to matrix $\boldsymbol{S} = \boldsymbol{S}_{\mathrm{cg}}$ (3.12) we can study systems associated with matrices $\tilde{\boldsymbol{S}} = \boldsymbol{S}_{\mathrm{be}}$ (3.11) and $\tilde{\boldsymbol{S}} = \boldsymbol{S}_{\mathrm{k}}$ (3.15). Here subscripts mean the type of Stäckel matrix for the different Drach systems.

The Stäckel systems with the constant matrix $\boldsymbol{S}_{\mathrm{cg}}$ possess the following 4×4 Lax matrices [15, 20]:

$$\mathcal{L}(\lambda) = \begin{pmatrix} L_1(\lambda, p_1, q_1) & 0 \\ 0 & L_2(\lambda, p_2, q_2) \end{pmatrix} \tag{5.2}$$

with independent 2×2 non-trivial blocks $L_j(\lambda)$. For instance, two standard blocks may be chosen

$$L_j(\lambda) = \begin{pmatrix} p_j & \lambda - q_j \\ -\left[\frac{\phi_j}{\lambda - q_j}\right]_{MN} & -p_j \end{pmatrix} \qquad L_j(\lambda) = \begin{pmatrix} \frac{p_j q_j}{\lambda} & 1 - \frac{q_j^2}{\lambda} \\ \frac{p_j^2}{\lambda} - \left[\frac{\phi_j}{1 - \frac{q_j^2}{\lambda}}\right]_{MN} & -\frac{p_j q_j}{\lambda} \end{pmatrix}.$$

Here $\phi(\lambda)$ is a parametric function on spectral parameter λ and $[\xi]_N$ are the linear combinations of the Laurent projections [15].

According to [16, 18], canonical transformations of the extended phase space induce a shift of the Lax matrices depending on the Hamilton function. Thus, using one known Lax matrix $\mathcal{L}(\lambda)$ (5.2) we can construct another Lax matrix. Namely, canonical transformations of the time (5.1) give rise to the following shift of the corresponding Lax matrices:

$$\tilde{\mathcal{L}}(\lambda) = \mathcal{L}(\lambda) - \tilde{H}\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \end{pmatrix} \qquad a, b = \pm 1 \qquad \text{or} \qquad \pm\,\mathrm{i} \tag{5.3}$$

where values of the constants a, b depend on the chosen form of the blocks $L_j(\lambda)$.

Below we present some Lax matrices constructed by the scheme designated above. In case (c) the Lax matrix is given by

$$\mathcal{L}_{\mathrm{c}}(\lambda) = \begin{pmatrix} p_1 & \lambda - q_1 & 0 & 0 \\ (\lambda + q_1)\left(-\frac{\alpha}{4} + \gamma q_1^2\lambda^2\right) & -p_1 & 0 & 0 \\ 0 & 0 & \mathrm{i}p_2 & \mathrm{i}(\lambda - q_2) \\ 0 & 0 & \mathrm{i}(\lambda + q_2)\left(-\frac{\alpha}{4} - \beta q_2^2\lambda^2\right) & -\mathrm{i}p_2 \end{pmatrix}$$

so the spectral curve

$$\Gamma(\lambda, \mu) : \det(\mathcal{L}_{\mathrm{c}}(\lambda) - \mu I) = 0$$

is a product

$$\left(\mu^2 - \frac{I_1}{2} - I_2 + \frac{\alpha\lambda^2}{4} + \frac{\gamma}{\lambda^2}\right)\left(\mu^2 - \frac{I_1}{2} + I_2 - \frac{\alpha\lambda^2}{4} + \frac{\beta}{\lambda^2}\right) = 0$$

of the corresponding canonical Stäckel curves (2.6).

In the case (k) the Lax matrix is given by

$$\tilde{\mathcal{L}}_{\mathrm{k}} = \begin{pmatrix} p_1 & \lambda - q_1 & 0 & 0 \\ -\frac{\gamma(\lambda+q_1)}{2} - \beta & -p_1 & 0 & 0 \\ 0 & 0 & \mathrm{i}p_2 & \mathrm{i}(\lambda + q_2) \\ 0 & 0 & \frac{\mathrm{i}\gamma(\lambda-q_2)}{2} + \mathrm{i}\beta & -\mathrm{i}p_2 \end{pmatrix} + \tilde{H}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}$$

where $\tilde{H} = I_1$ is the Hamilton function (3.9). As in the previous example the spectral curve

$$\left(\mu^2 - \frac{\gamma\lambda^2}{2} + (\beta + I_1)\lambda + \alpha + I_2\right)\left(\mu^2 + \frac{\gamma\lambda^2}{2} + (\beta - I_1)\lambda + \alpha - I_2\right) = 0$$

is a product of the corresponding Stäckel curves (2.6).

In case (b) the Lax matrix is given by

$$\tilde{\mathcal{L}}_{\mathrm{b}} = \begin{pmatrix} \frac{p_1q_1}{\lambda} & 1 - \frac{q_1^2}{\lambda} & 0 & 0 \\ \frac{p_1^2-(\beta-2\gamma)q_1^{-2}}{\lambda} & -\frac{p_1q_1}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{p_2q_2}{\lambda} & 1 + \frac{q_2^2}{\lambda} \\ 0 & 0 & \frac{-p_2^2+(\beta+2\gamma)q_2^{-2}}{\lambda} & -\frac{p_2q_2}{\lambda} \end{pmatrix} + \tilde{H}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $\tilde{H} = I_1$ is the Hamilton function (3.2). The corresponding spectral curve

$$\Gamma(y, \mu) : \det(\tilde{\mathcal{L}}_{\mathrm{b}}(\lambda) - yI) = 0$$

is a product

$$\left(y^2 - I_1 + \frac{2\alpha + I_2}{\lambda} - \frac{\beta + 2\gamma}{\lambda^2}\right)\left(y^2 - I_1 + \frac{2\alpha - I_2}{\lambda} - \frac{\beta - 2\gamma}{\lambda^2}\right) = 0$$

of the initial Stäckel curves (2.3), which could be rewritten in the canonical form (2.6).

All the spectral curves of these 4×4 Lax matrices $\mathcal{L}_c$, $\tilde{\mathcal{L}}_{\rm k}$ and $\tilde{\mathcal{L}}_{\rm b}$ give rise to the quadratic Stäckel integrals $I_{1,2}$ (2.1). The third integral K (2.7) may be extracted from the same matrices using multivariable universal enveloping algebras [15]. In fact, this integral is a coefficient of the following multivariable polynomial:

$$\operatorname{tr} P_\pi \mathcal{L}(\lambda_1) \otimes \mathcal{L}(\lambda_2) \otimes \mathcal{L}(\lambda_3) \otimes \mathcal{L}(\lambda_4) \tag{5.4}$$

where P_π is a permutation operator of auxiliary spaces corresponding to a Young diagram π [15]. The nonlinear algebra of integrals (2.8) may be reproduced using the Poisson bracket relations between the Lax matrices $\otimes_j^k \mathcal{L}(\lambda_j)$ [15]. The formulae (5.4) for the 256×256 matrices have been proved using the computer algebra system *Maple V*.

6. Conclusion

Let us discuss the list of all the known integrable natural Hamiltonian systems in the plane with a cubic integral [3, 5, 7]. We suppose that all these systems may be embedded into the family of the Stäckel systems [16]: either they may be embedded into the subset of the generalized Stäckel systems [18] or they may be related to the Toda lattices and the Calogero–Moser systems [5, 18]. As an example the last case (l) (3.10) of the Drach systems and the Fokas–Lagerstrom [4] model belong to the generalized Stäckel systems [18]. The complete classification will be presented in a forthcoming publication.

In this paper we proved this proposition for all the Drach systems [3]. Moreover, we rewrite the cubic integrals for the superintegrable Drach systems in the common form (2.7). This generalized 'angular momentum' may be used to construct another n-dimensional superintegrable Stäckel system with cubic integrals of motion. For instance, let us consider the Hamilton function

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{\gamma + \delta}{r} + \frac{1}{x^2 + y^2}\left(\frac{\alpha(r - z)}{r} + \frac{\beta(r + z)}{r} + U\left(\frac{y}{x}\right)\right) \tag{6.1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and $U(y/x)$ is an arbitrary function. The corresponding equations of motion are separable in the parabolic coordinates

$$q_1 = r + z \qquad q_2 = r - z \qquad q_3 = \arctan(y/x)$$

which are related to the following Stäckel matrix:

$$\boldsymbol{S} = \begin{pmatrix} 1 & 1 & 0 \\ q_1^{-1} & q_2^{-1} & 0 \\ q_1^{-2} & q_2^{-2} & -4 \end{pmatrix}.$$

The Hamilton function (6.1) coincides with the Stäckel integral I_1 (2.1) if

$$U_1 = \frac{\alpha}{q_1^2} + \frac{\gamma}{q_1} \qquad U_2 = \frac{\beta}{q_2^2} + \frac{\delta}{q_2} \qquad U_3 = U(q_3).$$

Thus we have an integrable Stäckel system with the independent integrals of motion $I_{1,2}$ and I_3, which are quadratic polynomials in momentum.

Canonical algebraic curves are defined in variables

$$\mu_1 = p_1 q_1 \qquad \mu_2 = p_2 q_2 \qquad \mu_3 = p_3.$$

To substitute these variables in the 'generalized angular momentum' (2.7) one obtains an additional integral of motion K cubic in momentum. In the initial physical variables this integral K looks like

$$K = (x^2 + y^2)p_z^3 - 2zp_z^2(p_x x + p_y y) - p_z(p_x x + p_y y)^2$$
$$+r^2 \left(\frac{\partial H}{\partial z}(p_x x + p_y y) + p_z \left(p_x^2 + p_y^2 - x\frac{\partial H}{\partial x} - y\frac{\partial H}{\partial y} \right) \right).$$

It will be interesting to understand the algebro-geometric origin of this 'generalized angular momentum' (2.7).

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